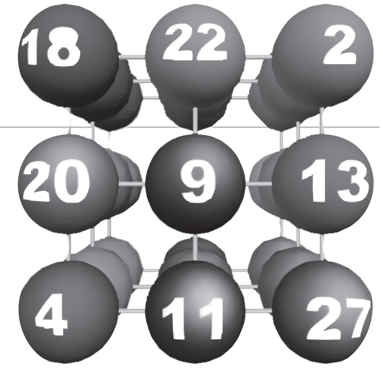
feature

Rotational sorcery

The inertial properties of magic squares and magic cubes



GRAPHIC BY ADAM ROGERS

BY **ADAM ROGERS** AND **PETER LOLY**

THE MOMENT OF INERTIA OF MAGIC SQUARES

Magic squares of order N are composed of the entries 1, 2, ... , N^2 arranged on a square unit lattice such that the sum of all entries along the rows, columns and main diagonals are equal to the magic constant of the square. An example of a magic square is shown below:

4	9	2
3	5	7
8	1	6

(1)

The magic constant can be easily found by summing the values 1, 2, ... , N^2 and dividing by N, the number of rows and columns to find

$$C_2 = \frac{N}{2}\left(N^2 + 1\right). \tag{2}$$

For $N = 3$, the magic constant is equal to 15. Though there is only one magic square of order 3 apart from trivial rotations and reflections of Equation (1), the number of squares per order quickly skyrockets. There are 880 distinct order-4 squares, and 275 305 224 distinct order-5 magic squares.

If we interpret magic squares as being composed of masses proportional to the entries of the squares, we can determine their moment of inertia about a given axis of rotation. The scalar moment of inertia I is found by summing $m_i r_i^2$ for each entry i, where r_i and m_i are the distance from the axis of rotation and the mass, respectively, of entry i. If we consider an axis of rotation through the middle row (in the case of even-order squares, the rotation axis lies between the two middle rows) and its counterpart through the middle column, it is obvious that the moments of inertia should be equal, since there are an equal number of rows/columns of the same total mass, each equally displaced from these axes. We can use the perpendicular axis theorem, which states:

$$I_z = I_x + I_y. \tag{3}$$

Since $I_x = I_y$, we have $I_z = 2I_x$. If we place an axis parallel to one edge of the square, it is easy to derive a general formula for the moment of inertia about that axis. Because we know the sum of values in a line and the spacing of the masses, we can find a formula in terms only of N. From here, we can use the parallel axis theorem to shift the axis so it passes through the centre of the square. Employing the perpendicular axis theorem, we find the simple formula:

$$I_z = \frac{1}{12}N^2\left(N^4 - 1\right). \tag{4}$$

For $N = 3$, $I_z = 60$, which can be verified explicitly using Equation (1). This is the only other property of magic squares, aside from the line sum, which is solely dependent on the order of the square, N. It is also worth noting that since we have only made use of the row and column line sums, the formula is general for semi-magic squares as well. These types of squares have only row and column line sums, but no restriction on the main diagonal sums. Equation (4) is consistent for large N with the moment of inertia of a continuous plate with mass $M = NC_2$ and $L = N$, reducing to $I = 1/6\,ML^2$. Because of the simple application of inertia principles and mathematics, this derivation is suitable for first-year physics students.

In addition to the derivations shown above, magic squares have a few practical applications, including uses in cryptography and image processing. When treated as matrices, magic squares also serve as exceptional examples of some advanced linear algebra theorems.

THE INERTIA TENSOR OF MAGIC CUBES

We can extend the concept of a magic square into the third dimension, yielding a magic cube. These cubes have constant Row, Column, Pillar (referred to as RCP) and main diagonal line sums. An example is presented below.

2	13	27
22	9	11
18	20	4

1st layer

16	21	5
3	14	25
23	7	12

2nd layer

24	8	10
17	19	6
1	15	26

3rd layer

(5)

Magic cubes are composed of entries 1, 2, ... , N^3, and in an analogous manner to the procedure for magic squares, we can find a formula for the line sum:

$$C_3 = \frac{N}{2}\left(N^3 + 1\right). \tag{6}$$

Since each of these layers is a magic square, though not of consecutive integers, it is easy to find the moment of inertia of a single layer as an extension of Equation (4), and thus, N stacked layers give the moment of inertia of a magic cube:

$$I_z = \frac{1}{12}N^3\left(N^3 + 1\right)\left(N^2 - 1\right). \tag{7}$$

By RCP symmetry, $I_x = I_y = I_z$. More formally, the inertia tensor is also diagonal (the off-diagonal elements vanish) with the origin of co-ordinates at the centre of the cube. This shows that magic cubes have the same inertial form as a spherical top.

CONCLUSIONS

Throughout our discussion of magic cubes, we have considered the entries in the magic cube only as masses. If we consider the entries instead as charges, we can neutralize the cube by subtracting the average entry from the rest of the cube. This will have the effect of causing the first three multipole moments of the charge distribution to vanish, with only off-diagonal elements of the octupole tensor remaining. To third order, a discrete charge distribution in the form of a magic cube will produce no electrical potential.

Just as magic squares are easily extended into the third dimension to create a magic cube, magic cubes can be extended further into the fourth dimension, forming magic hypercubes. These can, in fact, be generalized into N-dimensions, though these objects have not been the focus of our studies, since the inertia tensor and multipole expansion exist only in three dimensions.

For more information on magic squares, magic cubes and the inertia tensor, see:

A. Rogers and P. Loly. *American Journal of Physics*. (2004) **72**, pp. 786–789.

Clifford A. Pickover, *The Zen of Magic Squares, Circles, and Stars*. Princeton University Press. 2002.

Walter Trump, *http://www.trump.de/magic-squares*